Whitham deformations partially saturating the modulational instability in the nonlinear Schrodinger equation

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Abstract

In the framework of Gurevich and Pitaevskii approach [1] we construct modulated by Whitham [2] solution of nonlinear Shrodinger (NS) equation partially saturating the modulational instability. This solution describes new scenario of monochromatic wave evolution in NS equation which leads to generation of new phase and oscillation region.

1. From the point of view of Whitham deformation theory [2] modulational instability (see e.g. [3]) in the NS equation

$$iu_t + u_{xx} + 2 \mid u \mid^2 u = 0 \tag{1}$$

is connected with the complexity (i.e. non vanishing imaginary parts) of the characteristic speeds of the Whitham-NS equations. This leads to the exponential growth of the perturbations in the linearized (on the locally constant NS equation background) NS equation. It is natural to conjecture however that there are nontrivial (no constants) solutions of Whitham-NS system which have vanishing imaginary parts of some (not all!) characteristic speeds and hence partially saturating the modulational instability.

The purpose of this paper is to demonstrate on the simplest example the existence of the approximate solution of NS equation which partially saturates the modulational instability. This solution of (1) describes new (in contrast with ordinary modulational instability and some of its modifications [3-6]), nonlinear scenario of evolution of zero-phase solution (monochromatic wave) of NS equation

$$u_0(x,t) = \exp(i2t), \tag{2}$$

which leads to generation and propagation of new phase and oscillations in the (1) according to Whitham-NS equations. It is important that the characteristic development time for the new phase is the same order as the characteristic time of development of the ordinary modulational instability.

Evolution of the new phase leads to the formation of the oscillation region in the solution of (1). Due to the vanishing of one of the modulational instability increments in our case an analogy appears with the known [1] nature of the generation of new phases in modulational stable situations. Let us stress that physically principal difference of our situation from [1] is that the our scenario of generation of new phase which we propose below works (in contrast with [1]) in the case when asymptotics $u(x \to \pm\infty, t)$ of the solution $u(x,t)$ coincide.

2. The approximate solution of the NS equation which we propose here consists of the external zero-phase solution (2) in the region $x \leq x^-(t)$ and $x \geq x^+(t)$ and from the internal modulated by Whitham one-phase solution (see (3)-(13)) in the region $x^-(t) \leq x \leq x^+(t)$.

It is well known that one-phase solutions of (1) can be written in the form

$$u^\pm = \sqrt{f(\theta^\pm)} \cdot \exp(i\varphi^\pm), \quad \theta^\pm = x - U^\pm t, \tag{3}$$

$$f(\theta) = f_3 + (f_1 - f_3) dn^2\{\sqrt{f_1 - f_3} \cdot \theta; m\}, \quad m = (f_1 - f_2)/(f_1 - f_3), \tag{4}$$

$$\varphi_x^\pm = U^\pm/2 \mp A/f; \quad \varphi_t^\pm = -(U^\pm)^2/4 + (\sum_i f_i) \pm U^\pm A/f, \tag{5}$$

where $f_1 \geq f \geq f_2 \geq 0 \geq f_3$, $A = \sqrt{-f_1 f_2 f_3} \geq 0$, dn - is the Jacobi elliptic function. Elliptic spectral curve corresponding to (3)-(5) has branching points $(\lambda_2 = \lambda_1^*, \lambda_4 = \lambda_3^*)$ such that (c.f. [7]):

at upper sign:

$$\begin{array}{l} \lambda_1^+ \equiv \alpha^+ - i\gamma^+ = U^+/4 - \sqrt{-f_3}/2 - i(\sqrt{f_1} + \sqrt{f_2})/2, \\ \lambda_3^+ \equiv \beta^+ - i\delta^+ = U^+/4 + \sqrt{-f_3}/2 - i(\sqrt{f_1} - \sqrt{f_2})/2, \end{array} \tag{6}$$

at lower sign:

$$\begin{array}{l} \lambda_1^- \equiv \beta^- - i\delta^- = U^-/4 - \sqrt{-f_3}/2 - i(\sqrt{f_1} - \sqrt{f_2})/2, \\ \lambda_3^- \equiv \alpha^- - i\gamma^- = U^-/4 + \sqrt{-f_3}/2 - i(\sqrt{f_1} + \sqrt{f_2})/2, \end{array} \tag{7}$$

Whitham-NS equations for (1), (3)-(7) are (c.f. [7])

$$d\lambda_i/dt + S_i(\lambda) d\lambda_i/dx = 0, \quad i = 1,2,3,4, \tag{8}$$

$$\begin{array}{ll} S_1 = U + 2\lambda_{12}/(1 - \mu\lambda_{32}/\lambda_{31}), & S_3 = U + 2\lambda_{34}/(1 - \mu\lambda_{14}/\lambda_{13}), \\ S_2 = S_1^*, S_4 = S_3^*, & \mu \equiv E(m)/K(m), \end{array} \tag{9}$$

where $\lambda_{ij} \equiv \lambda_i - \lambda_j$, $m = \lambda_{21}\lambda_{43}/\lambda_{32}\lambda_{14}$, E, K - are the complete elliptic integrals of the second and first kind respectively.

In the oscillation region $0 \le x \le x^+(t)$, we use special solution of the Whitham-NS system (8)

$$\begin{array}{ll} \lambda_1^+ \equiv \alpha^+ - i\gamma^+ \equiv const.; & Im(S_3) = 0, \\ \{4\beta^+ + 2[(\gamma^+)^2 - (\delta^+)^2]/(\beta^+ - \alpha^+)\}t - x = g(\beta^+, \delta^+), & \end{array} \tag{10}$$

where $g(\beta, \delta)$ - is an arbitrary smooth function of its variables, which is determined from the initial conditions. Let us consider the simplest case $g \equiv 0$ (c.f. [1,8]). From the initial condition (2) we get $\gamma^+ \equiv 1$, $\alpha^+ \equiv 0$, and system (10) are invariant with respect to transformations $\delta^+ \to -\delta^+$ and $(\beta^+, x) \to -(\beta^+, x)$. Additional analysis shows that the system (10) is compatible and has unique solution with $\delta^+ \ge 0$, $\beta^+ \ge 0$, in the region $0 \le x \le x^+(t)$. Near the boundary $x^+ = x^+(t)$ the solution of the system (10) has form

$$\begin{array}{lll} x^+ = 4\sqrt{2}t, & x = x^+ - x', & 0 \prec x' \ll 1, \\ \beta^+ \approx 1/\sqrt{2} - 7x'/48t, & (\delta^+)^2 \approx x'/2\sqrt{2}t. & \end{array} \tag{11}$$

At the boundary $x = x^+(t)$ the solution u^+ from (3)-(5) is continuously glued with u_0 from (2). If $(x/t) \to +0$ the points $(\lambda_3^+, \lambda_4^+)$ closely come to the points $(\lambda_1^+, \lambda_2^+)$. In this limit our solution (3) degenerates into the soliton.

In the oscillation region $x^-(t) \le x \le 0$ solution of system (8) is defined by equations

$$\begin{array}{ll} \lambda_3^- \equiv \alpha^- - i\gamma^- \equiv const.; & Im(S_1) = 0, \\ \{4\beta^- + 2[(\gamma^-)^2 - (\delta^-)^2]/(\beta^- - \alpha^-)\}t - x = 0. & \end{array} \tag{12}$$

System (12) is analogous to system (10). From the initial condition (2) we obtain $\gamma^- \equiv 1$, $\alpha^- \equiv 0$. Near the boundary $x^- = x^-(t)$ solution of system (12) has form

$$\begin{array}{lll} x^- = -4\sqrt{2}t, & x = x^- + x'', & 0 \prec x'' \ll 1, \\ \beta^- \approx -1/\sqrt{2} + 7x''/48t, & (\delta^-)^2 \approx x''/2\sqrt{2}t. & \end{array} \tag{13}$$

At the boundary $x = x^-(t)$ the solution u^- from (3)-(5) is continuously glued with u_0 from (2). If $(x/t) \to -0$ the points $(\lambda_1^-, \lambda_2^-)$ closely come to the points $(\lambda_3^-, \lambda_4^-)$. In this limit our solution (3) degenerates into the soliton.

Due to Galilean invariance of (1) the above analysis may be easily extended to the case of the zero-phase solution (monochromatic wave)

$$u_0(x,t) = \exp[i2\alpha x + i(2 - 4\alpha^2)t].$$

Note the corresponding changes in the formulae (10)-(13): $x^\pm \to 4\alpha t + x^\pm$, $\alpha^\pm \to \alpha$, $\beta^\pm \to \alpha + \beta^\pm$, $\lambda_1^+ \to \alpha + \lambda_1^+$, $\lambda_3^- \to \alpha + \lambda_3^-$.

3. One - phase Whitham-NS equations (8) describe generally speaking two pairs of perturbations (corresponding to 4 branching points of our elliptic curve).

It seems natural to ask is it correct to "forget " about one pair of perturbations (note that we imposed restriction $\lambda_1^+ \equiv const.$ or $\lambda_3^- \equiv const.$) and investigate the above process in the "pure form"? The answer is that due to (11), (13) $\mid dx^{\pm}(t)/dt \mid \cong 1$, hence the characteristic time of the process considered above is of the same order as the characteristic time of development of modulational instability of the "forgotten "("thrown") mode (($\mid ImS_1^+ \mid, \mid ImS_3^- \mid) \cong 1$ at the initial time. Let us note also that $(\mid ImS_1^+ \mid, \mid ImS_3^- \mid)$ have maximal values in the external region and decrease (up to zero) by approaching to the center of the oscillation region.

Our consideration in this work are not rigorous. They are performed on the level of Gurevich and Pitaevskii [1] reasoning. Rigorous analysis of the Cauchy problem for (1), (2) requires much more sophisticated technique. One can get the impression of the corresponding difficulties on the simplest example of modulationally stable situation by comparing [1] with the papers [9] where justification of Gurevich and Pitaevskii conjecture (in the case of step-like initial data) was done.

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